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VIBRATIONS OF A STAMP ADHERING TO AN ELASTIC LAYER OVER PART OF ITS SURFACE

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The plane problem of harmonic vibrations of a stamp of finite size located in the segment $[a_1, a_3]$ on the surface of an elastic layer of thickness $2h$ is considered; the stamp adheres rigidly to the layer in the segment $[a_1, a_2]$ $(a_1 < a_2 < a_3)$ and makes contact without friction on the segment $[a_2, a_3]$.

A system of integral equations is constructed for the mixed problem with the necessary physical radiation principles taken into account, its unique solvability is established and an approximate method of solution is proposed. The method is based on applying factorization of functions and matrix-functions and permits reduction of the system of integral equations of the first kind to a system of integral equations of the second kind with a completely continuous operator in a certain function space, for which splitting into a finite and a small operator is effectively performed. This latter permits carrying out a finite approximation of the operator in a numerical application of the method.

The properties of the solutions of the integral equation, the contact stress distribution under the stamp, as well as the nature of the excited surface waves are studied. The presence of singularities in the contact stresses which hold not only on the stamp edge but also in the interior contact domain is established.

This problem is considered in connection with problems of defectoscopy of foundations by using vibration in order to clarify the nature of the contact with an elastic medium, as well as with the defectoscopy of bonded fits.

The realization of the most complex situation occurring when using the method applied, the approximate factorization of a specific matrix-function, is presented as an illustration.

Static problems for stamps with incomplete separation of the boundary conditions were considered in [1, 2] for somewhat different purposes.

1. To reduce the problem under consideration to integral equations, we solve the dynamic Lamé equations (with inertial terms) in the domain $|z| \leqslant h, -\infty < x < \infty$ under the following boundary conditions:

$$\begin{aligned}
&z = -h, \ u = w = 0, \ -\infty < x < \infty \\
&z = h, \ u = u_0(x), \ x \in [a_1, a_2] \\
&w = w_0(x), \ x \in [a_1, a_3] \\
&\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0, \quad x \in (-\infty, a_1) \cup (a_2, \infty)
\end{aligned}$$

$$\lambda \frac{\partial u}{\partial x} + (\lambda + 2\mu) \frac{\partial w}{\partial z} = 0, \quad x \in (-\infty, a_1) \cup (a_3, \infty)$$

$$(U(x, z, t) = \mathrm{Re}\,[u(x, z)\, e^{-i\omega t}], \ W(x, z, t) = \mathrm{Re}\,[w(x, z)\, e^{-i\omega t}])$$

Here U and W are, respectively, the tangential and normal displacements in the layer λ, μ are Lamé coefficients, h is half the layer thickness, and ω is the frequency of stamp vibrations.

Using the physical principle of limiting absorption [3,4], the problem formulated above is reduced to a system of integral equations of the form

$$\sum_{s=1}^{2} \int_{a_1}^{a_{s+1}} r_{ks}(x - t)\, q_s(t)\, dt = f_k(x), \quad x \in [a_1, a_{k+1}] \tag{1.1}$$

$$r_{mn}(t) = \int_{\Gamma} R_{mn}(u)\, e^{iut}\, du \tag{1.2}$$

by the Fourier transform method.

The elements of the matrix $\mathbf{R}(u)$ are connected to the elements of the real matrix $\mathbf{K}(u)$ by the relationships

$$\begin{aligned}
&R_{mm} \equiv K_{mm}, \ R_{12} \equiv -R_{21} \equiv iK_{12} \\
&K_{11} = 1/2\, \varkappa_2^2 (\sigma_2 \operatorname{sh} 2\sigma_2 \operatorname{ch} 2\sigma_1 - \sigma_1^{-1} u^2 \operatorname{sh} 2\sigma_1 \operatorname{ch} 2\sigma_2)\, \Delta^{-1}(u) \\
&K_{12} = -u\, \{(2u^2 - \varkappa_2^2/2)(1 - \operatorname{ch} 2\sigma_1 \operatorname{ch} 2\sigma_2) + \sigma_1^{-1}\sigma_2^{-1} \times \\
&\quad [2u^4 - u^2 (3/2\, \varkappa_2^2 + \varkappa_1^2) + \varkappa_1^2 \varkappa_2^2] \operatorname{sh} 2\sigma_1 \operatorname{sh} 2\sigma_2\}\, \Delta^{-1}(u) \\
&K_{22} = 1/2\, \varkappa_2^2 (\sigma_1 \operatorname{sh} 2\sigma_1 \operatorname{ch} 2\sigma_2 - \sigma_2^{-1} u^2 \operatorname{sh} 2\sigma_2 \operatorname{ch} 2\sigma_1)\, \Delta^{-1}(u) \\
&\Delta(u) = u^2 (2u^2 - \varkappa_2^2) - (2u^4 - u^2 \varkappa_2^2 + 1/4\, \varkappa_2^4) \operatorname{ch} 2\sigma_1 \operatorname{ch} 2\sigma_2 + \\
&\quad \sigma_1^{-1}\sigma_2^{-1} u^2\, [2u^4 - u^2 (2\varkappa_2^2 + \varkappa_1^2) + \varkappa_1^2 \varkappa_2^2 + 1/4\, \varkappa_2^4] \times \\
&\quad \operatorname{sh} 2\sigma_1 \operatorname{sh} 2\sigma_2 \\
&\varkappa_1^2 = \omega^2 \rho h^2 (\lambda + 2\mu)^{-1}, \ \varkappa_2^2 = \omega^2 \rho h^2 \mu^{-1}, \ \sigma_k = (u^2 - \varkappa_k^2)^{1/2}
\end{aligned} \tag{1.3}$$

Here ν, ρ are the Poisson's ratio and the density of the material, $q_1(x)$, $q_2(x)$ are, respectively, the tangential and normal contact stresses, $f_1(x)$, $f_2(x)$ are given amplitudes of the tangential and normal displacements of points under the stamp, respectively, taken with the factor $4\pi\mu h$.

The functions $K_{mn}(u)$ in (1.3) are regular everywhere on the real axis with the exception of the same poles for all the functions $\pm \zeta_k$ $(k = 1, 2, \ldots, p)$, K_{mm} are even, K_{12} is odd. As $|u| \to \infty$ the elements of the matrix-function $\mathbf{K}(u)$ have the form

$$K_{mm}(u) = A\, |u|^{-1} [1 + O(u^{-1})], \ K_{12}(u) = Bu^{-1} \times [1 + O(u^{-1})], \ A > |B|$$

The contour Γ in (1.2) is disposed in conformity with the rules set up in [4-6].

Distribution curves of the real poles of elements of the matrix $\mathbf{K}(u)$ in (1.3) are presented in Fig. 1 for $\nu = 0.3$. It is seen that the number of poles increases with the increase in frequency (which corresponds to a growth of the parameter $\varkappa_2$).

2. At this time no theory has been developed for systems of integral equations of the form (1.1). Hence, we present one of the uniqueness criteria for the solution of the system of integral equations (1. 1) in the space $L_\alpha(a_1, a_3)$, $\alpha > 1$, which was successfully set up and which always assures uniqueness in specific problems in case the system degenerates into one equation.

Theorem 1. Let functions $K_{mn}(u)$ which have single poles, possess the following properties:

$$1)\ [K_{11}^{-1}(\zeta_r)]' > 0,\quad r = 1, 2, \ldots, p$$
$$2)\ [K_{11}^{-1}(\zeta_r)]'\,[K_{22}^{-1}(\zeta_r)]' - \{[K_{12}^{-1}(\zeta_r)]'\}^2 > 0$$

3) There exist rational functions $P_{mn}(u)$ bounded at infinity, with poles at the points $\pm\zeta_k$ $(k = 1, 2, \ldots, p)$ such that the real Hermitian component of the matrix $\mathbf{R}(u)\,\mathbf{P}^{-1}(u)$ is positive definite for any u $(-\infty < u < \infty)$.

Using the method proposed in [4] under the assumption of single poles of the functions $K_{mn}(u)$ for the proof, we multiply the first equation of the system by $\bar{q}_1(x)$, and the second by $q_2(x)$ and integrate over the whole axis for $f_k(x) = 0$ (the bar denotes the complex conjugate). We add the results obtained and deform the contour of integration to the real axis. After extraction of the real part and equating it to zero, we conclude on the basis conditions 1), 2) that

$$Q_k(\zeta_r) \equiv Q_k(-\zeta_r) \equiv 0,\qquad r = 1, 2, \ldots, p;\quad k = 1, 2$$
$$Q_k(u) = \int_{a_1}^{a_{k+1}} q_k(x)\, e^{iux}\, dx$$

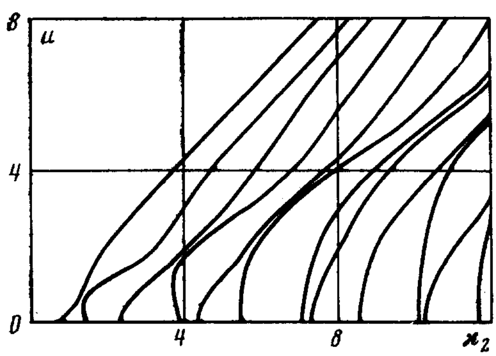

Fig. 1

Introducing the functions

$$T_n(u) = \sum_{s=1}^{2} P_{ns}(u)\, Q_s(u) \qquad (2.1)$$

which are the Fourier transforms of functions from L_α which vanish outside $[a_1, a_{n+1}]$, as has been established in [5], and determining $Q_s(u)$ from (2. 1), we insert them into (1. 1). Subsequently, the requirement of positive-definiteness of the operator in the left side of (1. 1) results in conditions 3).

3. Let us reduce the system of integral equations of the first kind to a system of integral equations of the second kind with a completely continuous operator. Let us apply to this end the method of factorization of functions and matrix-functions. The factorization method permits satisfaction of the discontinuous boundary conditions of the boundary value problem.

The boundary value problem has three points of change of boundary conditions: at the point a_1 two boundary conditions change at once, hence, the matrix-function is factorized here, and at the points a_2 and a_3 just one boundary condition changes. The functions should be factorized at the points mentioned.

To realize these proposals, we apply a complex Fourier transform to the system (1. 1). The system consequently takes the form

$$\mathbf{R}(u)\,\mathbf{Q}(u) = \mathbf{F} + e^{ia_1u}\mathbf{\Phi}^- + e^{ia_2u}\mathbf{\Psi}^+ \tag{3.1}$$

$$\mathbf{Q}(u) = \{Q_1, Q_2\}, \quad Q_1^- = Q_1 e^{-ia_2u}$$

$$\mathbf{\Phi}^- = \left\{ \int_{-\infty}^{a_1} \varphi_1 e^{iu(x-a_1)}dx, \int_{-\infty}^{a_1} \varphi_2 e^{iu(x-a_1)}dx \right\} = \{\Phi_1^-, \Phi_2^-\} \tag{3.2}$$

$$\mathbf{F} = \left\{ \int_{a_1}^{a_2} f_1 e^{iux}dx, \int_{a_1}^{a_3} f_2 e^{iux}dx \right\} = \{F_1, F_2\} \tag{3.3}$$

$$\Psi_k^+ = \int_{a_{k+1}}^{\infty} \psi_k e^{iu(x-a_{k+1})}dx, \quad \mathbf{\Psi}^+ = \{\Psi_1^+, e^{iu(a_3-a_2)}\Psi_2^+\}$$

Here φ_1, φ_2 is a continuation of the right side of the integral equations (1.1) in the domain $x < a_1$. Analogously ψ_1, ψ_2 is the continuation of the right sides of these equations in the domains $x > a_2$ and $x > a_3$, respectively.

Having determined these functions, we find the solution $Q(u)$ of the system of integral equations from the relationship (3.1).

Assuming the existence of a solution $\mathbf{q}(x) = \{q_1, q_2\}$ in the class L_α, it can be established that the functions φ_k, ψ_k are bounded at infinity as $|x| \to \infty$ by degenerating into trigonometric polynomials representing waves being radiated. They are differentiable in any bounded set.

In connection with the above the Fourier transforms $\mathbf{\Phi}^-$, $\mathbf{\Psi}^+$ exist on the contour Γ as limits of the appropriate integrals (3.2) and (3.3), respectively, from the lower and upper half-planes.

The relationship (3.1) will be equivalent to the system of integral equations (1.1) if it is considered on the contour Γ ; in its neighborhood this contour determines a certain curvilinear strip Ω of the regularity of all the functions of (3.1). Henceforth, all the functions will be factorized relative to contours located in Ω, particularly with respect to the contour Γ.

We shall assume the following factorizations to be satisfied:

$$\mathbf{R}(u) = \mathbf{K}_+\mathbf{K}_- = \mathbf{N}_-\mathbf{N}_+ \tag{3.4}$$

$$K_{11} + K_{21}K_{12}/K_{22} \equiv C(u) = C_+C_- \tag{3.5}$$

$$K_{22}(u) = \Pi_+\Pi_-$$

Let us later introduce the following notation for the function $g(u)$ which is regular in Ω and decreases by a power-law at infinity:

$$g(u) = \{g\}^+ + \{g\}^-$$

$$\{g\}^+ = \frac{1}{2\pi i}\int_{\Gamma_-} \frac{g(z)}{z-u}dz, \qquad u \text{ above } \Gamma_-$$

$$\{g\}^- = -\frac{1}{2\pi i}\int_{\Gamma_+} \frac{g(z)}{z-u}dz, \qquad u \text{ below } \Gamma_+$$

The contours Γ_- and Γ_+ lie, respectively, below and above the contour Γ in Ω.

As is known, the operations $\{\ \}^+$, $\{\ \}^-$ denote the projection of an analytic function in the region above (E_+) and below (E_-) the contour Γ, respectively.

Now we multiply the relationship (3.1) by the matrix $e^{-iua_1}\mathbf{K}_-^{-1}$ and we project the result on E_-. We arrive at a relationship of the form

$$\mathbf{K}_-^{-1}\mathbf{\Phi}^- + \{e^{-iua_1}\mathbf{K}_-^{-1}\mathbf{F}\}^- + \{\mathbf{K}_-^{-1}e^{iu(a_2-a_1)}\mathbf{\Psi}^+\}^- = 0 \qquad (3.6)$$

Eliminating the function Q_2 from the system (3.1), dividing the result by $e^{ia_2u}C_+$ and after making a projection on E_+, E_-, we obtain, respectively,

$$\frac{\Psi_1^+}{C_+} + \left\{\frac{F_1}{C_+}e^{-ia_2u}\right\}^+ + \left\{\frac{e^{-i(a_2-a_1)u}}{C_+}\Phi_1^-\right\}^+ - \{DF_2e^{-ia_2u}\}^+ - \qquad (3.7)$$
$$\{De^{-i(a_2-a_1)u}\Phi_2^-\}^+ - \{De^{i(a_3-a_2)u}\Psi_2^+\}^+ = 0$$
$$Q_1^-C_- - \left\{\frac{F_1}{C_+}e^{-ia_2u}\right\}^- - \left\{\frac{e^{-i(a_2-a_1)u}}{C_+}\Phi_1^-\right\}^- +$$
$$\{De^{-i(a_2-a_1)u}\Phi_2^-\}^- + \{De^{i(a_3-a_2)u}\Psi_2^+\}^- + \{DF_2e^{-ia_2u}\}^- = 0$$
$$(D(u) = iK_{12}/(C_+K_{22}))$$

Finally, dividing the second equation of the system (3.1) by $e^{iua_3}\Pi_+$ and projecting the result on E_+, we obtain a relationship of the form

$$\frac{\Psi_2^+}{\Pi_+} + \left\{\frac{e^{-i(a_3-a_1)u}}{\Pi_+}\Phi_2^-\right\}^+ + \left\{\frac{e^{-ia_3u}F_2}{\Pi_+}\right\}^+ - \left\{\frac{iK_{21}}{\Pi_+}e^{-i(a_3-a_2)u}Q_1^-\right\}^+ = 0 \qquad (3.8)$$

Let us introduce new unknowns by setting

$$\mathbf{X}^- = \mathbf{K}_-^{-1}\mathbf{\Phi}^- = \{X_1^-, X_2^-\}, \quad \mathbf{X}^+ = \mathbf{\Psi}^+\Pi_+^{-1} = \{X_3^+, X_4^+\}$$
$$Q_1^-C_- = X_5^-$$

The relationships (3.6)–(3.8) are a closed system relative to the unknowns $X_k^\pm$. Let us reduce this system to equations of the second kind with a completely continuous operator. To this end, let us represent the system in the form

$$\mathbf{X}^- + \{\mathbf{K}_-^{-1}e^{iu(a_2-a_1)}\Pi_+\mathbf{X}^+\}^- + \{e^{-iua_1}\mathbf{K}_-^{-1}\mathbf{F}\}^- = 0 \qquad (3.9)$$
$$\frac{\Pi_+}{C_+}X_3^+ + \{De^{i(a_3-a_2)u}X_4^+\Pi_+\}^- - De^{iu(a_3-a_2)}X_4^+\Pi_+ +$$
$$\left\{\frac{e^{-i(a_2-a_1)u}}{C_+}(k_{11}X_1^- + k_{12}X_2^-)\right\}^+ -$$
$$\{De^{-i(a_2-a_1)u}(k_{21}X_1^- + k_{22}X_2^-)\}^+ + \left\{\left(\frac{F_1}{C_+} - DF_2\right)e^{-ia_2u}\right\}^+ = 0$$
$$X_5^- - \frac{e^{-i(a_2-a_1)u}}{C_+}\left[X_1^-\left(k_{11} - \frac{iK_{12}}{K_{22}}k_{12}\right) + \right.$$
$$\left. X_2^-\left(k_{21} - \frac{iK_{12}}{K_{22}}k_{22}\right)\right] + \left\{\frac{e^{-i(a_2-a_1)u}}{C_+}(k_{11}X_1^- + k_{12}X_2^-)\right\}^+ -$$
$$\{De^{-i(a_2-a_1)u}(k_{21}X_1^- + k_{22}X_2^-)\}^+ + \{De^{i(a_3-a_2)u}X_4^+\Pi_+\}^- -$$
$$\left\{\left(\frac{F_1}{C_+} - DF_2\right)e^{-ia_2u}\right\}^- = 0$$
$$X_4^+ + \left\{\frac{e^{-i(a_3-a_1)u}}{\Pi_+}(k_{12}X_1^- + k_{22}X_2^-)\right\}^+ +$$
$$\left\{\frac{iK_{21}}{C_-\Pi_+}e^{-i(a_3-a_2)u}X_5^-\right\}^+ + \left\{\frac{e^{-ia_3u}}{\Pi_+}F_2\right\}^+ = 0$$

Here k_{mn} are elements of the matrix $\mathbf{K}_-(u)$.

Transferring all the integral terms of the system (3.9) to the right side and solving the linear system in the left side for $X_k^{\pm}$, we arrive at a system of the second kind with a completely continuous operator.

4. Under the assumption of existence of a solution we establish the properties of the solutions of the system (3.9) and of the system (1.1) as well. We shall consider the right sides $f_1(x)$ and $f_2(x)$ of the integral equations (1.1) to belong to C_2 in $[a_1, a_2]$ and $[a_1, a_3]$, respectively. In this case, certain continuations of the functions f_1, f_2 have Fourier transforms F_1, F_2 which decrease at infinity as u^{-2} [7]. The behavior of the free terms of the system (3.9) is set in the neighborhood of infinity on any contour in the region Ω by using this estimate. Namely, the free terms on any of the mentioned contours belong to the space $c(1)$; here $c(\lambda)$ is the space of functions which is continuous with the weight $|z|^{\lambda}$ on the contour.

Studying (3.9) in the space $c(\lambda), \lambda \leqslant 1$, we conclude that if solutions of the system exist, then they possess the properties: $X_k^{\pm} \in c(1)$. Hence, $\Phi^-, \Psi^+ \in c(^3/_2)$; $Q_1^- \in c(^1/_2)$. Using these properties, we obtain the following series representations of the functions in the neighborhood of infinity by using (3.6) (3.8):

$$\Phi_k^-(z) = \frac{c_1(k)\, e^{ia_1 z}}{z^{3/2+i\varepsilon}} + O(z^{-5/2})$$

$$\Psi_k^+(z) = \frac{c_2(k)\, e^{ia_{k+1} z}}{z^{3/2}} + O(z^{-5/2})$$

$$Q_1(z) = \frac{c_3 e^{ia_1 z}}{z^{1/2}} + \frac{c_4 e^{-ia_2 z}}{z^{1/2+i\varepsilon}} + O(z^{-3/2})$$

$$Q_2(z) = \frac{c_5 e^{ia_1 z}}{z^{1/2+i\varepsilon}} + \frac{c_6 e^{i(a_2+a_3) z}}{z^{1/2}} + O(z^{-3/2})$$

Applying the inverse Fourier transform, we establish the following properties for the functions q_k on the basis of these estimates:

$$\begin{aligned} &q_1(x)(x-a_1)^{1/2+i\varepsilon}(a_2-x)^{1/2} \in C(a_1, a_2) \\ &q_2(x)(x-a_1)^{1/2+i\varepsilon}\,|a_2-x|^{1/2}(a_3-x)^{1/2} \in C(a_1, a_3) \end{aligned} \tag{4.1}$$

i.e., $q_k \in L_\alpha$ the uniqueness space. Hence, single-valued solvability of the system (3.9) and therefore of the system (1.1) in the class (4.1) follows for arbitrary right sides $f_1, f_2 \in C_2$.

5. To construct an approximate solution of the system of integral equations (3.9), let us note the following: since the operators on the left side are completely continuous Fredholm operators, then the solution of the system can be written in the form of a Fredholm series by using the apparatus of exterior analysis [8]. Truncating this series yields an approximate solution of the problem. The complexity in using this approach is the problem of evaluating the multiple integrals.

Another approximate approach is to reduce the Fredholm system to an algebraic system of linear equations [9] in conjuction with an approximate method of factorizing the functions and matrix-functions. It is hence assumed that by realizing the approximate factorization of the functions and matrix-functions by using irrational functions, a natural approximation of the Fredholm integral operators by finite-dimensional operators is carried out. This approximation is realized by dropping the contours of integration in the integral operators $\{\ \}^+$ and raising the contours in the operators $\{\ \}^-$.

In the process of these operations, when the poles intersect during contour motion, splitting of the operator into a finite-dimensional operator and a small operator occurs (smallness is achieved by remoteness of the branch point to which the contour is shifted, and by the presence of the damping exponentials). Neglecting the small operator, we obtain the finite-dimensional approximation.

Let us demonstrate this with an example of the last equation in the system (3.9). Let λ_{1k} $(k = 1, 2, \ldots, r_1)$, λ_{2k} $(k = 1, 2, \ldots, r_2)$, λ_{3k} $(k = 1, 2, \ldots, r_3)$ denote, respectively, the poles of the functions $k_{12}\Pi_+^{-1}$, $k_{22}\Pi_+^{-1}$ and $iK_{21}\Pi_+^{-1}$. We obtain

$$X_4^+(z) - \sum_{k=1}^{r_1} \frac{e^{-i(a_3-a_1)\lambda_{1k}}}{\lambda_{1k} - z} \operatorname*{Res}_{z=\lambda_{1k}} (k_{12}\Pi_+^{-1}) X_1^-(\lambda_{1k}) - \tag{5.1}$$

$$\sum_{k=1}^{r_2} \frac{e^{-i(a_3-a_1)\lambda_{2k}}}{\lambda_{2k} - z} \operatorname*{Res}_{z=\lambda_{2k}} (k_{22}\Pi_+^{-1}) X_2^-(\lambda_{2k}) -$$

$$\sum_{k=1}^{r_3} \operatorname*{Res}_{z=\lambda_{3k}} (iK_{21}\Pi_+^{-1}) X_5^-(\lambda_{3k}) \frac{e^{-i(a_3-a_2)\lambda_{3k}}}{(\lambda_{3k} - z) C_-(\lambda_{3k})} + D_4(z) = 0$$

Here $D_4(z)$ is the approximation of the last term on the left side of the equation under consideration. The remaining equations of the system (3.9) can be transformed in an analogous manner.

It is seen from (5.1) that it is sufficient to know the values X_1^-, X_2^-, X_5^- at the points λ_{ik} $(i = 1, 2, 3)$, respectively, for an approximate determination of $X_4^+(z)$. Setting $z = \lambda_{ik}$ in the appropriate equations of the system obtained from (3.9) by the method described above, and acting in a similar manner, we obtain a closed system of linear algebraic equations by starting from the necessity to determine $X_k^\pm$ $(k = 1, 2, 3, 5)$. Having determined $X_k^\pm$, we find Φ^- and Ψ^+ and then $\mathbf{Q}$ from (3.1).

To investigate the behavior of the surface outside the stamp, it is sufficient to evaluate the complex Fourier inversion for the functions Φ^- and Ψ^+.

The behavior of the layer surface in the far zone is computed sufficiently simply in connection with the following representation of the functions Φ^- and Ψ^+:

$$\Phi_n^- = \sum_{k=1}^{p} \frac{c_{kn}}{z - \zeta_k} + \chi_{1n}(z), \quad \Psi_n^+ = \sum_{k=1}^{p} \frac{d_{kn}}{z + \zeta_k} + \chi_{2n}(z)$$

Here the functions χ_{1n}, χ_{2n} are regular in the domains $\operatorname{Im} z \leqslant \delta_{1n}$ and $\operatorname{Im} z \geqslant \delta_{2n}$ $(\delta_{kn} > 0)$, respectively. We obtain the following representations for $\varphi_k^-(x)$, $\psi_k^+(x)$ as a result of the inversion:

$$\varphi_k^-(x) = 2\pi i \sum_{s=1}^{p} c_{sk} e^{-i\zeta_s x} + O(e^{\delta_{1k} x}), \quad x \to -\infty$$

$$\psi_k^+(x) = -2\pi i \sum_{s=1}^{p} d_{sk} e^{i\zeta_s x} + O(e^{-\delta_{2k} x}), \quad x \to \infty$$

6. The most complicated part in the application of this approximate method is the construction of an approximate factorization of the functions and the matrix-functions. After execution of this problem, the further solution is not difficult.

In conformity with the general theorems elucidated in [10], the conditions imposed above on the matrix $\mathbf{R}(u)$ assure its factorization into the form (3.4). However, it is impossible to obtain an explicit form for the matrices $\mathbf{K}_{\pm}(u)$ and $\mathbf{N}_{\pm}(u)$. Hence, we apply the approximate factorization [11], which is realized only by using a digital computer.

The following theorem, whose proof based on the method of perturbations is omitted for the sake of brevity, is the basis for the application of approximate factorization of the matrix-functions.

Theorem 2. Let the system of equations (1.1) with the matrix-kernel $\mathbf{K}(u)$ have the solution $\mathbf{q} = \{q_1, q_2\}$, and with the matrix $\mathbf{K}^*(u)$ the solution $\mathbf{q}^* = \{q_1^*, q_2^*\}$. If the elements $K_{ij}(u)$ and $K_{ij}^*(u)$ of the matrices $\mathbf{K}(u)$ and $\mathbf{K}^*(u)$ satisfy the conditions

$$|K_{ij}(u) - K_{ij}^*(u)|\,|K_{ij}(u)|^{-1}(1 + |u|)^{\alpha} < \varepsilon, \quad \alpha > 1/2$$

under the conditions of Theorem 1, then the inequality

$$|[q_k(x) - q_k^*(x)]\sqrt{|a_3 - x|\,|a_2 - x|\,|a_1 - x|}| < m\varepsilon$$

is valid for sufficiently small ε, where m is independent of q_k.

Let us introduce the functionally commutative matrix $\mathbf{G}(u)$ with elements $G_{mn}(u)$ of the form $G_{11} = G_{22} = K_{11}, G_{12} = -G_{21} = iK_{12}$. The matrix $\mathbf{G}(u)$ is factorized explicitly [11]

$$\begin{aligned}
&\mathbf{G}(u) = \mathbf{G}_+(u)\,\mathbf{G}_-(u) \equiv \mathbf{G}_-(u)\,\mathbf{G}_+(u) \qquad (6.1)\\
&2G_{11}^{\pm}(u) = 2G_{22}^{\pm}(u) = s_{\pm}(u) + t_{\pm}(u), \quad 2G_{12}^{\pm}(u) = \\
&\quad -2G_{21}^{\pm}(u) = i[t_{\pm}(u) - s_{\pm}(u)]\\
&s_+s_- = K_{11} - K_{12}, \quad t_+t_- = K_{11} + K_{12}
\end{aligned}$$

We factorize (6.1) approximately. We examine the factorization of a function $S(u) = K_{11} + K_{12}$ by using an approximating function $S^*(u)$ selected to accuracy ε from the condition

$$|S(u) - S^*(u)| / |S(u)| < \varepsilon \qquad (6.2)$$

Let $z_s^+ > 0$ $(s = 1, 2, \ldots, n_1)$ and $z_k^- < 0$ $(k = 1, 2, \ldots, n_2)$ are the zeroes of the function $S(u)$. Without limiting the generality, we can consider $n_1 = n_2 = n$, where $n \leqslant p$. We form a function which has no real zeroes and poles

$$S_1(u) = S(u)(A_1^2 + u^2)^{-N} R_1(u)\, U_1(u) \qquad (6.3)$$

Here

$$R_1(u) = \left(\frac{u^2+b^2}{A^2-B^2}\right)^{1/2} \exp\left(-\frac{1}{\pi}\ln\frac{A+B}{A-B}\operatorname{arctg}\frac{u}{b}\right) \times$$

$$\prod_{s=1}^{N}(u^2+\xi_s^2)\prod_{k=1}^{p}(u^2-\zeta_k^2)\prod_{s=1}^{n}(u-z_s^+)^{-1}(u-z_s^-)^{-1}$$

$$U_1(u) = \begin{cases}\prod\limits_{s=1}^{p-n}(u^2+v_s^2)^{-1}, & n < p\\ 1, & n = p\end{cases}$$

Here $b \gg 1$ in (6.3) is a previously assigned parameter of the approximation. Together with the parameter $A_1 > 0$, the quantities $\xi_k > 0$ $(k = 1, 2, \ldots, N)$ and $v_k > 0$ $(k = 1, 2, \ldots, p - n)$ are additional parameters of the approximation which satisfy condition (6.2) of least deviation of $S^*(u)$ from $S(u)$ on the real axis; $S_1(u) \sim 1$ as $|u| \to \infty$.

For uniqueness of the function $S_1(u)$ we draw a slit from bi to $i\infty$ and from $-bi$ to $-i\infty$ and we fix the branch by the condition $\sqrt{b^2} = b > 0$. After the substitution $x = u^2/(u^2 + A_1^2)$, which converts the positive half-axis into the segment $[0, 1]$, we approximate the function $S_1(A_1\sqrt{x/(1-x)})$ by Bernshtein polynomials $B_N(x)$ of degree $2N$. We obtain

$$S^*(u) = T_{2N}(u)\,[R_1(u)]^{-1}\,U_1^{-1}(u) \tag{6.4}$$

since $B_N(u^2/(u^2 + A_1^2))$ is a rational function, whose denominator is the polynomial $(u^2 + A_1^2)^N$ and the numerator is a polynomial $T_{2N}(u)$ of order $2N$.

Now taking into account that

$$\exp\left(\frac{1}{\pi}\ln\frac{A+B}{A-B}\operatorname{arctg}\frac{u}{b}\right) = \left(\frac{b+iu}{b-iu}\right)^{i\gamma}; \quad \gamma = -\frac{1}{2\pi}\ln\frac{A+B}{A-B}$$

$S^*(u)$ can be factorized explicitly. The function $K_{11} - K_{12}$ is factorized analogously.

Let us introduce the matrix

$$\mathbf{H}(u) = \mathbf{G}_-^{-1}(u)\,\mathbf{R}(u)\,\mathbf{G}_+^{-1}(u) \tag{6.5}$$

Its elements have the form

$$H_{11} = 1 + \theta(K_{11} - L^+), \quad H_{22} = 1 + \theta(K_{11} + L^+)$$
$$H_{12} = -i\theta(K_{12} + L^-), \quad H_{21} = i\theta(K_{12} - L^-)$$
$$(\theta = {}^1\!/_2(K_{22} - K_{11})/\det\mathbf{G}(u), \quad L^\pm = {}^1\!/_2(s_+t_- \pm s_-t_+))$$

If the roots of the approximating polynomial $T_{2N}(u)$ equal $\alpha_k, \bar{\alpha}_k$ $(k = 1, 2, \ldots, N)$, then we obtain on the basis of (6.4)

$$L^\pm(u) = \frac{1}{2}\left(\frac{A^2 - B^2}{u^2 + b^2}\right)^{1/2}\prod_{k=1}^{p}(u^2 - \zeta_k^2)^{-1}\prod_{k=1}^{N}(u^2 + \xi_k^2)^{-1}\times$$
$$\left\{\prod_{s=1}^{n}[u^2 - (z_s^+)^2]\prod_{k=1}^{N}(u^2 - \alpha_k^2)(b^2 + u^2)^{i\gamma} \pm \prod_{s=1}^{n}[u^2 - (z_s^-)^2]\times\right.$$
$$\left.\prod_{k=1}^{N}(u^2 - \bar{\alpha}_k^{-2})(b^2 + u^2)^{-i\gamma}\right\}U_1^{-1}(u)$$

Hence, it is seen that if there are no real roots for the function $S(u)$ or they are located symmetrically relative to the origin, then $H_{11}(u)$ and $H_{22}(u)$ are real on the real axis and $H_{12}(u)$ and $H_{21}(u)$ are complex conjugates.

The relationships

$$H_{mm} = 1 + O(u^{-1}), \quad H_{mn} = O(u^{-1}), \quad m \neq n$$

evidently hold as $|u| \to \infty$.

Let us approximate the elements of the matrix $\mathbf{H}(u)$ by rational functions by using Bernshtein polynomials according to the scheme represented above for the function $S(u)$. We shall hence select the approximation parameters from the condition of smallness of the difference between the absolute values of the matrix elements of $\mathbf{H}(u)$ and the approximating functions with respect to $|\det \mathbf{H}(u)|$ on the real axis. We obtain the matrix $\mathbf{H}^*(u)$ with elements of the form

$$H_{ij}^*(u) = V_{ij}(u) \prod_{s=1}^{N_1} \frac{1}{u^2+\eta_s^2} \prod_{k=1}^{m_1} \frac{1}{u^2-t_k^2} \tag{6.6}$$

Here η_s are approximation parameters, and t_k are positive poles of elements of the matrix $\mathbf{H}(u)$ (the same for all the elements). The degree of the polynomials $V_{11}(u)$, $V_{22}(u)$ is one higher than the degree of the polynomials $V_{12}(u)$ and $V_{21}(u)$.

To factorize $\mathbf{H}^*(u)$ it is sufficient to factorize the matrix $\mathbf{V}(u)$ with the elements $V_{ij}(u)$. As is known [12], the matrix $\mathbf{V}(u)$ can be represented in the form

$$\mathbf{V}(u) = \mathbf{X}(u)\,\mathbf{W}(u)\,\mathbf{Y}(u) \tag{6.7}$$

where $\mathbf{X}(u)$ and $\mathbf{Y}(u)$ are polynomial matrices with constant nonzero determinants, and $\mathbf{W}(u)$ is a diagonal matrix factorizable explicitly.

Therefore, taking (6.5)-(6.7) into account, we find

$$\mathbf{N}_-(u) = \prod_{k=1}^{N_1} \frac{1}{u-\eta_k} \prod_{k=1}^{m_1} \frac{1}{u-t_k} \mathbf{G}_-(u)\,\mathbf{X}(u)\,\mathbf{W}_-(u)$$

$$\mathbf{N}_+(u) = \prod_{k=1}^{N_1} \frac{1}{u+\eta_k} \prod_{k=1}^{m_1} \frac{1}{u+t_k} \mathbf{W}_+(u)\,\mathbf{Y}(u)\,\mathbf{G}_+(u)$$

The second approximate factorization of (3.4) is constructed analogously.

As an illustration, let us consider the approximate factorization of the matrix $\mathbf{R}(u)$ for $\varkappa_2=1.25$, $b=10.0$, $\nu=0.3$. In this case there exists one positive pole $p_1=0.5759$ for the matrix elements of $\mathbf{R}(u)$.

The function $S(u)=K_{11}(u)+K_{12}(u)$ is approximated well by using the eighth power polynomial $T(u)$ with the roots: $0.689 \pm 1.034i$, $-0.254 \pm 0.854i$, $-0.069 \pm 2.186i$, $0.478 \pm 7.437i$. For the approximation $v_1=0.5$, $\xi_1=1.20$, $\xi_2=1.25$, $\xi_3=1.00$, $\xi_4=3.50$, $A_1=1.100$.

The auxiliary matrix $\mathbf{H}(u)$ has the positive pole $t_1=0.4995$. The elements $H_{11}(u)$ and $H_{22}(u)$ have no real zeros, but there is one zero $z_1=0.7043$ on the positive half-axis for the element $H_{12}(u)$.

In approximating the elements of the matrix $\mathbf{H}(u)$ by using fourth power polynomials, the error in approximating $H_{11}(u)$ does not exceed 8%, 2% for $H_{22}(u)$ and 11% for $H_{12}(u)$. Hence $\eta_1=0.05$, $\eta_2=1.20$. Consequently, we obtain the diagonal matrix $\mathbf{W}(u)$ whose diagonal elements are: unity and the polynomial $w(u)$, where

$$w(u) = \prod_{k=1}^{6} (u^2-\alpha_k^2)$$

$$\alpha_1=0.075,\ \alpha_2=0.445i,\ \alpha_3=0.146+0.073i$$

$$\alpha_4=0.146-0.073i,\ \alpha_5=1.208i,\ \alpha_6=1.447i$$

The polynomial matrices $\mathbf{X}(u)$ and $\mathbf{Y}(u)$ have determinants equal to unity. The elements X_{mn} have the form

$$X_{11}=u^6+1.67\ u^4+2.38\cdot10^{-2}u^2+0.719\cdot10^{-4}$$
$$10^{-5}X_{12}=0.613iu^5-0.137u^4+1.02iu^3-0.230u^2+0.0152iu-0.00548$$
$$X_{21}=-0.186iu^5+0.313u^4+0.0804iu^3-0.158u^2+0.00582iu+0.00159$$
$$10^{-5}X_{22}=0.114u^4+0.217iu^3-0.0919u^2-0.108iu+0.0176$$

The matrix $\mathbf{Y}(u)$ has the elements

$$Y_{11}=Y_{22}=1,\quad Y_{21}=0$$
$$10^{-5}Y_{12}=-0.613iu^{11}-0.0711u^{10}-2.80iu^9+0.167u^8-3.32iu^7+0.637u^6-$$
$$-0.350iu^5+0.0833u^4+0.0106iu^3-0.00196u^2-0.300\cdot10^{-3}iu+0.375\cdot10^{-4}$$

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